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Bounded natural frequencies of completely free rectangular plates

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Abstract

This paper presents dimensionless natural frequency parameters for completely free rectangular plates, calculated using the superposition method (SM) and the finite difference method (FDM). The convergence tests show that while the SM gives upper bound results the FDM gives lower bound results. The results obtained using the FDM appears to be the best lower bound results available to date. Together with the upper bound results from the SM, the maximum possible error in the first 12 non-zero natural frequency parameters for completely free rectangular plates of various aspects ratios ranging from 1:1 to 3:1 can be determined from the results presented here.

1. Introduction

The free vibration analysis of beams and plates has a long research history, and some of this work was devoted to estimation of upper and lower bounds to the natural frequencies. An excellent review of the literature relating to vibration analysis of plates and the methods used was published by Leissa [1]. Most of these methods give upper bounds for the natural frequencies of the completely free plates as the solution is either based on assumed shapes which effectively overconstrain the system or using the superposition [2] of exact solutions for plates with more constraints. Among the methods for finding upperbounds, Gorman's superposition method (SM) [2,3] is very efficient and appears to give the lowest (and therefore the best) upperbound values for the natural frequencies of plates with various aspect ratios. However, it gives upperbound only for certain boundary conditions. Of the articles that give both upper and lower bound results, Ku [4] gives upper and lower bounds for the fundamental natural frequency of beams by using the Rayleigh quotient and Timoshenko quotient, respectively. These methods are not applicable for determining higher modes. Marangoni et al. [5] present bounded natural frequencies of clamped orthotropic rectangular plates; they obtained the upperbound results using the Rayleigh–Ritz method and the lower bound results using a decomposition method. Kuttler and Sigillito [6] give estimates of upper and lower bounds for frequencies of trapezoidal and triangular clamped plates by using selected trial functions.

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Nomenclature

a	plate dimension in x direction
b	plate dimension in y direction
D	plate flexural rigidity, $(Eh/12)/(1-\nu^2)$
E	modulus of elasticity of material
H	mesh size parallel to x direction, $a/(P-1)$
K	mesh size parallel to y direction, $b/(Q-1)$
m	mass per unit area of plate
M	bending moment distributed along edge of plate
P	the number of nodes in x direction
Q	the number of nodes in y direction
V	plate vertical edge reaction
w	plate lateral deflection
x, y	plate spatial coordinates
Φ	aspect ratio of plate b/a
ν	Poisson's ratio of material
Ω	$\omega a^2\sqrt{\rho/D}$
ω	radian frequency of vibration

The search for a method to find a lower bound solution for completely free plates led to several articles on the well-known finite difference method (FDM). It is worth noting here that in 1943, Courant [7], regarded the FDM as often preferable to other methods, such as the Rayleigh–Ritz method. In 1956, Weinberger [8] has shown that the FDM gives lower bounds for Laplace equations with constrained boundaries for first natural frequency and subsequently gave a proof showing that this is true for higher modes also [9]. In Ref. [8], Weinberger states that although the focus of the paper is for an eigenvalue problem involving a Laplace operator, the lowerboundedness is true for higher order operators such as the biharmonic operator too. This means that the FDM may be expected to give lower bound results for eigenvalues of plates. Hubbard has then shown that the FDM gives lower bound for natural frequencies for a free membrane [10]. The proofs for the lowerboundedness of the natural frequencies are based on the central difference formula, although this is not spelled out in these references. However, the references make it clear that for the lower boundedness to hold, the FD grid area must include the entire physical system. The authors are unaware of any specific proof for the lowerboundedness of the natural frequencies of completely free plates. However, the nature of convergence of the numerical results for the natural frequencies of completely free rectangular plates indicates that the lowerboundedness holds for plates too.

The results presented here were obtained by solving the plate vibration equation using the FDM and the SM for the case of completely free rectangular plates with various aspect ratios.

2. Procedure

2.1. Governing differential equation

The partial differential equation governing the out-of-plane vibration of rectangular plates is [11]

$$\frac{\partial^4 W(x,y)}{\partial x^4}+2\frac{\partial^4 W(x,y)}{\partial x^2\partial y^2}+\frac{\partial^4 W(x,y)}{\partial y^4}-\frac{m\omega^2}{D}W(x,y)=0. \tag{1}$$

For the FDM, the plate is meshed as shown in Fig. 1(a) and Eq. (1) is approximated in the finite difference form with central-difference approximation, in terms of the nodal displacements. The molecular FD equation at node (i,j) depends on the values of displacement of this and adjacent nodes shown in Fig. 1(b). The basic finite difference operators are given below.

$$\left(\frac{\partial w}{\partial x}\right)_{i,j}=\frac{w_{i+1,j}-w_{i-1,j}}{2H}, \tag{2}$$

$$\left(\frac{\partial w}{\partial y}\right)_{i,j}=\frac{w_{i,j+1}-w_{i,j-1}}{2K}, \tag{3}$$

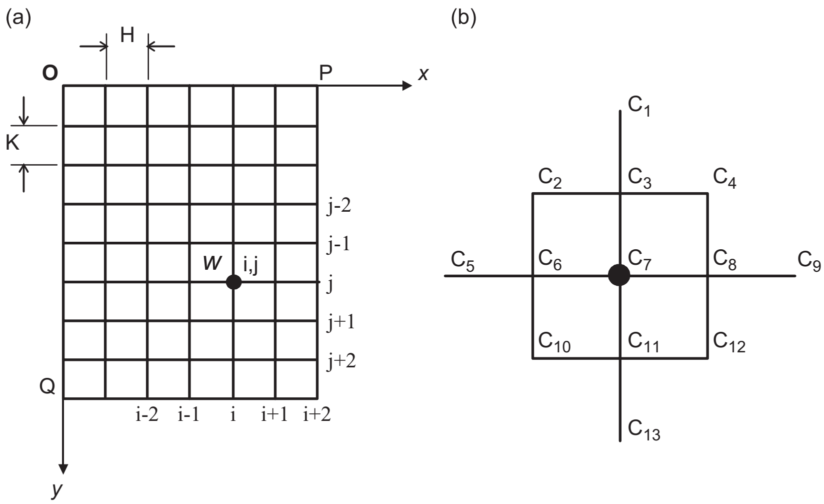

Fig. 1. A meshed completely free plate (a), and FD molecular formula of the plate governing equation (b).

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_{i,j} = \frac{w_{i-1,j} - 2w_{i,j} + w_{i+1,j}}{H^2}, \tag{4}$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{i,j} = \frac{w_{i,j-1} - 2w_{i,j} + w_{i,j+1}}{K^2}, \tag{5}$$

$$\left(\frac{\partial^2 w}{\partial x \partial y}\right)_{i,j} = \frac{w_{i-1,j-1} - w_{i+1,j-1} - w_{i-1,j+1} + w_{i+1,j+1}}{2HK}, \tag{6}$$

$$\left(\frac{\partial^3 w}{\partial x^3}\right)_{i,j} = \frac{-w_{i-2,j} + 2w_{i-1,j-1} - 2w_{i+1,j} + w_{i+2,j}}{2H^3}, \tag{7}$$

$$\left(\frac{\partial^3 w}{\partial x^2 \partial y}\right)_{i,j} = \frac{w_{i-1,j+1} - 2w_{i,j+1} + w_{i+1,j+1} - (w_{i-1,j-1} - 2w_{i,j-1} + w_{i+1,j-1})}{2H^2K}, \tag{8}$$

$$\left(\frac{\partial^3 w}{\partial y^3}\right)_{i,j} = \frac{-w_{i,j-2} + 2w_{i,j-1} - 2w_{i,j+1} + w_{i,j+2}}{2K^3}, \tag{9}$$

$$\left(\frac{\partial^3 w}{\partial x \partial y^2}\right)_{i,j} = \frac{w_{i+1,j-1} - 2w_{i+1,j} + w_{i+1,j+1} - (w_{i-1,j-1} - 2w_{i-1,j} + w_{i-1,j+1})}{2HK^2}, \tag{10}$$

$$\left(\frac{\partial^4 w}{\partial x^4}\right)_{i,j} = \frac{w_{i-2,j} - 4w_{i-1,j} + 6w_{i,j} - 4w_{i+1,j} + w_{i-2,j}}{H^4}, \tag{11}$$

$$\left(\frac{\partial^4 w}{\partial y^4}\right)_{i,j} = \frac{w_{i,j-2} - 4w_{i,j-1} + 6w_{i,j} - 4w_{i,j+1} + w_{i,j+2}}{K^2}, \tag{12}$$

$$\left(\frac{\partial^4 w}{\partial x^2 \partial y^2}\right)_{i,j} = \frac{w_{i-1,j-1} - 2w_{i,j-1} + w_{i+1,j-1} - 2(w_{i-1,j} - 2w_{i,j} + w_{i+1,j}) + w_{i-1,j+1} - 2w_{i,j+1} + w_{i+1,j+1}}{H^2K^2}. \tag{13}$$

The FD form of Eq (1) is

$$\sum_{k=1}^{13} C_k w_k^* = 0, \tag{14}$$

where

$$\begin{aligned} &w_1^* = w(i,j-2),\quad w_2^* = w(i-1,j-1),\quad w_3^* = w(i,j-1),\\ &w_4^* = w(i+1,j-1),\quad w_5^* = w(i-2,j),\quad w_6^* = w(i-1,j),\quad w_7^* = w(i,j),\\ &w_8^* = w(i+1,j),\quad w_9^* = w(i+2,j),\quad w_{10}^* = w(i-1,j+1),\\ &w_{11}^* = w(i,j+1),\quad w_{12}^* = w(i+1,j+1),\quad w_{13}^* = w(i,j+2), \end{aligned} \tag{15}$$

$$C_1 = C_{13} = \frac{1}{K^4}, \tag{16}$$

$$C_5 = C_9 = \frac{1}{H^4}, \tag{17}$$

$$C_2 = C_{12} = \frac{2}{H^2K^2}, \tag{18}$$

$$C_4 = C_{10} = \frac{2}{H^2K^2}, \tag{19}$$

$$C_3 = C_{11} = -4\left(\frac{1}{K^4} + \frac{1}{H^2K^2}\right), \tag{20}$$

$$C_6 = C_8 = -4\left(\frac{1}{H^4} + \frac{1}{H^2K^2}\right), \tag{21}$$

$$C_7 = \left(\frac{6}{H^4} + \frac{6}{K^4} + \frac{8}{H^2K^2}\right) - \frac{m\omega^2}{D}. \tag{22}$$

The central difference formula reduces the error in the second derivatives to the order of H^4, K^4 and the error in the higher derivatives would also be of this or higher order.

2.2. Boundary conditions

Figs. 2(a) and (b) show the distribution of nodes when the FD equation is applied at an edge of the plate and at a corner, respectively, where w_1^* to w_{13}^* are nodal deflections. The projected deflections outside of the plate need to be expressed in term of the deflections at the internal nodes by using boundary conditions.

The boundary conditions of a free edge are given in Leissa's monograph [1]. Bending moment and vertical edge reaction at a free edge are zero. Those boundary conditions at the edges are expressed as:

at $x = 0$ or a ($i = 1$ or P),

$$\left(\frac{\partial^2 w}{\partial x^2}\right) + \nu\left(\frac{\partial^2 w}{\partial y^2}\right) = 0, \tag{23a}$$

$$\left(\frac{\partial^3 w}{\partial x^3}\right) + \nu^*\left(\frac{\partial^3 w}{\partial x \partial y^2}\right) = 0, \tag{23b}$$

at $y = 0$ or b ($j = 1$ or Q),

$$\left(\frac{\partial^2 w}{\partial y^2}\right) + \nu\left(\frac{\partial^2 w}{\partial x^2}\right) = 0, \tag{24a}$$

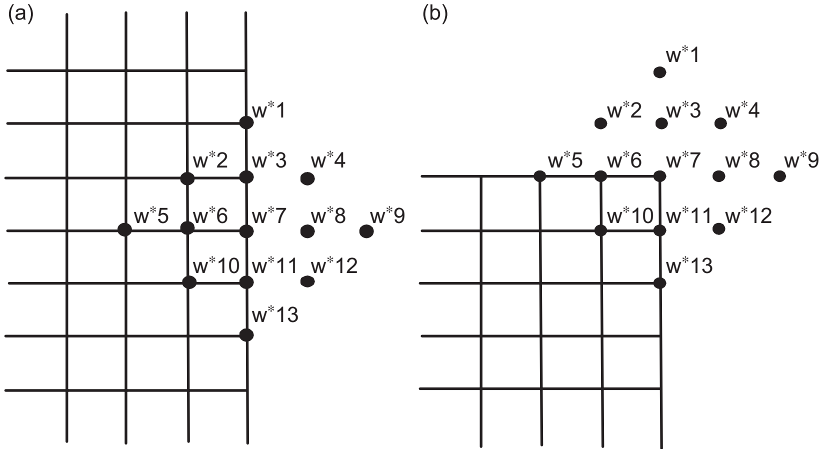

Fig. 2. The node distribution (a) at the edge and (b) at the corner.

$$\left(\frac{\partial^3 w}{\partial y^3}\right) + \nu^* \left(\frac{\partial^3 w}{\partial x^2 \partial y}\right) = 0. \tag{24b}$$

The above boundary conditions are not enough to cover all nodes outside of the plate at the corners. The following boundary condition at the corners [1], are also needed.

$$\left(\frac{\partial^2 w}{\partial x \partial y}\right) = 0 \quad \text{(at the corners)}. \tag{25}$$

3. Result and discussion

The numerical results obtained using the FDM and the SM are presented and compared to the natural frequency parameters in earlier literature. The natural frequencies of completely free plates are given in the dimensionless form, $\Omega = \omega a^2 \sqrt{\rho/D}$, which will be referred to as the natural frequency parameter. The natural frequency parameter is importantly related to plate aspect ratio ($\Phi = b/a$) rather than the dimensions of plate. All natural frequency parameters were calculated by using the software MATLAB in default double precision. However, the maximum number of nodes used in the FDM to compute the natural frequency parameters is limited to 55×55 due to computing limitations.

Table 1 shows the non-zero natural frequency parameters of completely free rectangular plates with aspect ratios ranging from 1:1 to 3:1 obtained by the FDM. The two letters adjacent to the values indicate the type of modal shapes. SS, AA, SA and AS mean that the mode is symmetric about both the x- and y-axes, antisymmetric about both axes, symmetric about the x-axis and antisymmetric about the y-axis, and antisymmetric about the x-axis and symmetric about the y-axis, respectively. The natural frequency parameters obtained by using the SM are also shown in Table 1. The corresponding results obtained by using the FDM have excellent agreement with results of the SM. The results by the FDM are slightly lower than those by the SM.

Convergence test were carried out for all the above aspect ratios and modes. The results for the fundamental natural frequency of a square plate are shown in Fig. 3. It can be seen that both methods give converging results; the FDM gives lower bound results while the SM gives upper bound results. The rate of convergence of the SM is significantly better than that of the FDM. The first author generated the results for a free plate using the SM [12] for various aspect ratios and number of terms. The results for other aspects ratios and for higher modes are not presented in this paper but they are available in the ME Thesis of the first author [12], which also gives the MATLAB codes developed for this work. The results in Ref. [12] show that both SM and FDM consistently give results that converge towards each other (upper and lower bounds) as the matrix size is

Table 1
The lower and upper bounds of natural frequency parameters for various aspect ratios for $\nu = 0.3$

Φ	1			1.25			1.5		
Mode	Lower bound (FDM)	Upper bound (SM)		Lower bound (FDM)	Upper bound (SM)		Lower bound (FDM)	Upper bound (SM)	
1	13.46	13.47	AA	10.75	10.76	AA	8.926	8.931	AA
2	19.57	19.60	SS	13.57	13.59	SS	9.503	9.517	SS
3	24.24	24.27	SS	22.36	22.39	SS	20.57	20.60	SA
4	34.75	34.80	SA or AS	25.86	25.89	SA	22.15	22.18	SS
5	34.75	34.80		30.34	30.38	AS	25.58	25.65	AS
6	60.90	61.09	SA or AS	39.34	39.45	AS	29.73	29.79	AS
7	60.90	61.09		50.15	50.30	AA	38.05	38.16	AA
8	63.56	63.69	SS	51.39	51.49	SS	43.84	43.93	SS
9	69.04	69.27	AA	60.74	60.94	SA	53.07	53.35	SS
10	76.95	77.17	AA	69.27	69.49	AA	59.82	60.05	SA
11	105.1	105.5	SA or AS	76.17	76.58	SS	64.63	64.92	SA
12	105.1	105.5		80.23	80.48	AS	65.55	65.75	AS
Φ	2			2.5			3		
1	5.358	5.366	SS	3.428	3.433	SS	2.379	2.382	SS
2	6.640	6.644	AA	5.275	5.278	AA	4.373	4.375	AA
3	14.60	14.62	SA	9.511	9.541	AS	6.596	6.617	AS
4	14.85	14.90	AS	11.31	11.33	SA	9.233	9.244	SA
5	21.97	22.00	SS	18.53	18.63	SS	12.96	13.03	SS
6	25.31	25.38	AA	18.87	18.92	AA	15.04	15.07	AA
7	25.96	26.00	AS	22.41	22.45	SS	21.16	21.31	AS
8	29.53	29.68	SS	24.40	24.44	AS	22.19	22.23	SS
9	35.97	36.04	SS	28.62	28.75	SA	22.19	22.29	SA
10	39.86	40.05	SA	31.38	31.45	SS	24.29	24.35	AS
11	48.16	48.45	AS	31.39	31.63	AS	28.60	28.67	SS
12	50.33	50.58	AS	40.96	41.22	AS	31.02	31.23	AA

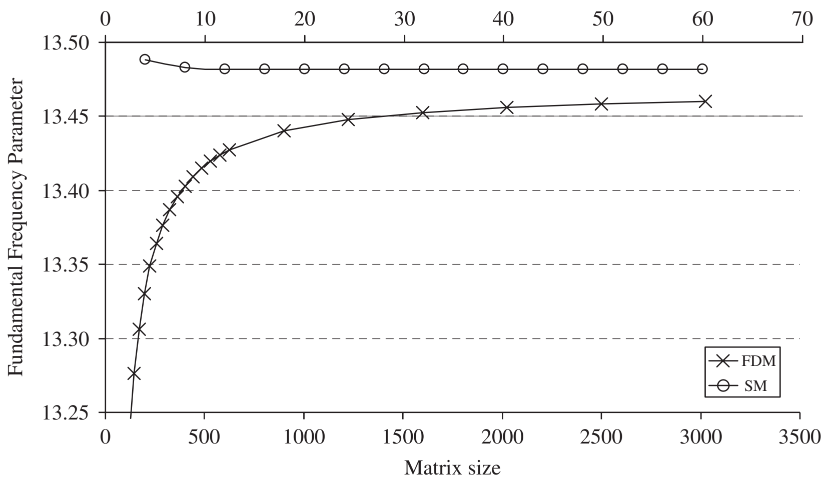

Fig. 3. Fundamental natural frequency parameters for a square plate by the FDM and the SM.

increased. As noted in the case of the fundamental natural frequency, the rate of convergence of the SM proved to be always very good and was significantly faster than that of the FDM but for all the cases tested, the convergence of the FDM was from below while the SM converged from above.

Table 2
Comparison of natural frequency parameters obtain by FDM with those in Leissa's publication [1] for the doubly antisymmetric modes of the square free plate ($v = 0.3$)

Present	Leissa	
	Lower bound	Upper bound
$b/a = 1$		
13.46	13.092	13.474
69.04	66.508	69.576
76.95	75.146	77.411
$b/a = 1.25$		
10.75	10.479	10.761
50.15	48.352	50.487
69.27	67.665	69.746
$b/a = 1.5$		
8.926	8.6667	8.9351
38.05	36.651	38.294
66.50	64.844	66.965
$b/a = 2$		
6.640	6.4563	6.6464
25.31	24.417	25.455
58.32	56.151	59.051

The work shows that Gorman's SM gives excellent convergence in its results for the fundamental natural frequency parameters with only 20 terms [2]. These results also confirm the prediction in a recent publication [3] that gives a mathematical proof that the application of Gorman's SM [2,11] would give upper bound results for a free plate. No proof for the free plate appears to exist to the authors' knowledge (it is proved that the FDM gives lower bounds for the natural frequencies in only specific cases [8–10]), but the results show that they are lower bounds for the natural frequencies of the completely free plates.

Although, exact results for completely free rectangular plates are not available, calculated values of the natural frequencies confirm that the natural frequencies decrease with number of terms used in the SM and increase with number of nodes used in the FDM. Thus the exact natural frequencies of a completely free plate are bracketed by the results of the SM and the FDM.

In Table 2 the natural frequency parameters obtained by using the FDM are compared with the results published in Leissa's review [1]. The upper bound and lower bound results by Leissa were taken from Ref. [13]. It is seen that the results of the FDM are lower than published upper bounds but are higher than the previously published lower bounds and are very close to the upper bounds indicating these may be the best lower bound solutions available to date.

The natural frequency parameters in Table 1 give these bounds for completely free rectangular plates. These appear to be the lowest upper bound and highest lower bound solutions available to date. Since the rate of convergence of the SM is significantly faster, the upperbound results may be regarded as bench marks while the difference between these and the lower bound results may be used to determine the maximum possible error due to the discretisation.

4. Conclusion

The natural frequencies of completely free plates with various aspect ratios were computed by using the SM and the FDM. The results by the FDM appear to be the best lower bounds for the natural frequencies of completely free rectangular plates available so far, and there is excellent agreement between these and the upperbound values found in earlier literature. The results seem to converge to exact natural frequencies of the plates as the mesh size approaches to zero. The SM converges very quickly, significantly faster than the FDM.

The maximum number of nodes used to compute the natural frequency parameters is limited to 55×55 due to computing limitations. It is expected that the discrepancy between the FDM and SM results could be narrowed if the eigensolver is modified or if computer memory is increased. The work also shows that Gorman's SM gives excellent convergence in its results for the fundamental natural frequencies with only 20 terms.

The SM results may be regarded as bench marks results for completely free rectangular plates, although they may not be exact. The FDM results presented in this paper together with the SM results give an estimate of the maximum possible error in these bench mark values.

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